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Bound States of Monopoles and Dyons with Spin-1/2 Particles in SU(2) Gauge Theory Under Moduli Space Approximation

S.C. Joshi · R. Swarup

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Abstract Energy spectrum and degeneracy associated with bound states of monopoles and dyons in non-Abelian gauge theory has been investigated and it is shown that energy levels expand due to the presence of additional degeneracies. Splitting of energy level of dyonium in presence of external magnetic and electric field has also been analyzed confirming the presence of additional degeneracy levels of the system. In addition, the study of behaviour of a fermion moving in the field of non-Abelian dyon in moduli space under SU(2) gauge potential has been undertaken and energy eigen values for the system are carried out. Detailed analysis of relativistic correction in fermion-dyon system in moduli space is presented and angular momentum operators of the system are derived, which demonstrates that besides the contribution of Higgs field, the interaction of spin and orbital angular momentum of moving fermion also contributes to the energy operator.

Keywords Dyon · Dyonium · Degenerate states · Spin–orbit interaction · Perturbation · Moduli space · Gauge transformation · Relativistic approximation · Four-potential · Orbital angular momentum operator

1 Introduction

Two significant advancements have been reported in the theory of magnetic monopoles one is the quantum field theory of interacting electric and magnetic charges as developed by Schwinger [1, 2] and the other is the local Lagrangian quantum field theoretic formulation developed by Zwanzigar [3, 4]. In previous approaches, Schwinger postulated both the Hamiltonian and commutation rules, which while successful, seems an adhoc method of

S.C. Joshi (✉)
Department of Physics, Ahlcon International School, Mayur Vihar, Phase 1, Delhi 91, India
e-mail: Scj.2007@rediffmail.com

R. Swarup
Department of Physics, D.S. College, Aligarh, U.P., India
e-mail: rsdsc_alg@yahoo.com

formulating a quantum field theory. Zwanzigar's formulation requires a great deal of clarification on canonical and physical structure basis of the theory.

In Yang–Mills vacuum there are two types of topological excitations; instantons and monopoles. Instantons and monopoles interact coherently in the θ vacuum [5–13] and this might carry the solution of the strong CP problem [14–16]. Monopoles in pure Abelian gauge theory are the manifestations of instanton excitation, gives rise dyon [11–13, 17–19]. Furthermore, we can say that theoretical instantons containing topological charges are made of dyons. Schwinger [1, 2], however, ignored the experimental failures to detect monopole as well as the theories casting doubts on its existence, formulated a relativistically covariant quantum field theory of monopoles and sharpened Dirac's quantization condition by restricting the product of electric and magnetic charges to integer values. This quantization condition explains to some extent the negative experimental results of monopole search but in this theory the rotational symmetry was violated due to the presence of singular lines in the solution of vector potential around a monopole. Subsequently, it became clear that the monopole and dyon can be understood better in non-Abelian gauge theories [20–23]. Perhaps the reason for our inability to see these particles with certainty lies elsewhere rather than in their inconsistencies of relativistic quantum field theory. Julia and Zee [24–28] extended the idea of 't Hooft–Polyakov to construct classical solutions for non-Abelian dyons which arise as quantum mechanical excitations of fundamental monopoles. They come automatically from the semi classical quantization of global-charge-rotation degrees of freedom of monopoles.

In this paper, we present a rigorous study of non-Abelian gauge theory (can be cast into Abelian gauge theory with monopoles) for dyons which proves that the instanton-anti-instanton bound state may be the possible spring regions of dyon. This may be possible only if instantons and (anti)-instantons are strongly correlated. The angular momentum operator and degree of degeneracy of the system has been carried out showing that monopole loop and the loop of electric currents fall on the top of each other. Further, we investigate the behaviour of dyonium in presence of external magnetic and electric field to demonstrate that the energy levels of the system are modified due to the presence of magnetic charge on dyon. Thereafter energy levels of the system are lifted causing the splitting in the usual way.

Extending this work in the third section, the study of interaction of spin-$1/2$ particles (fermion) in the field of a non-Abelian dyon in moduli space has been undertaken and Dirac equations for the system are derived. Extra spin contribution in the energy gained by the fermion in the field of non-Abelian dyon is studied. In fourth section, motion of fermion in generalized electromagnetic field of non-Abelian dyon with spin–orbit interaction has been undertaken that shows besides the contribution of Higgs field, the interaction of spin and orbital angular momentum of moving fermion also contributes to the energy operator which arise as relativistic interactions in different parts of the Hamiltonian.

2 Phenomenological Behaviour of Bound States of Monopoles and Dyons in External Fields Under SU(2) Gauge Potential

Now it has been widely recognized [24–28] that SU(5) grand unified model [24–28] is a gauge theory that contains monopole and dyon solutions. Consistently monopoles and dyons have become an intrinsic part of all current grand unified theories [24–28]. The bound state of two dyons consist different solutions than the bound state of hydrogen atom due to the presence of magnetic charge on dyon. The study of energy spectrum and degeneracies associated with harmonic oscillator has been carried out and it has been shown that the degeneracy of the level E_n are $(n+1)(n+2)/2$. Dynamical symmetry group of a three dimensional harmonic oscillator of monopole is SU(3). By using inverse transformations of raising and lowering operators [20–23] the angular momentum operator of the system can be written as

$$\begin{aligned} L_j &= (\vec{r} \times \vec{p})_j \\ &= \frac{i\hbar}{2} \sum_{k,l=1}^{3} \varepsilon_{ijk} (a_k a_l^\dagger - a_l a_k^\dagger), \end{aligned} \tag{2.1}$$

where a and $a^\dagger$ are raising and lowering operators of the eigen values of Hamiltonian of the system. Bound states of instantons–(anti-) instanton system thus provides very small Bohr radius as the value of magnetic charge is very large in comparison of electric charge [20–23, 29, 30]. n^2 fold degeneracy of the levels of this system is confirmed by making identification $n = 2a + 1$ of the energy eigen value

$$E = -\left(\frac{\mu g^4}{2\hbar^2 (2a+1)^2} \right), \tag{2.2}$$

where μ is the reduced mass of the system.

Integer value of the topological charge of the instantons may be given as

$$q = \frac{1}{32\pi^2} \int d^4x \, (\partial_\mu a_\nu - \partial_\nu a_\mu) \varepsilon_{\mu\nu\rho\sigma} F_{\rho\sigma} \in \mathbb{Z}. \tag{2.3}$$

The gauge degrees of freedom of Carton subgroup U(1) $\subset$ SU(2) are kept dynamical so that the monopoles may arise. Monopoles [5–10, 31, 32] are arise by mapping gluonic degrees of freedom of the theory onto photons, colour electric charges and colour magnetic charges, which is identified as Abelian projection.

Conserved electric current of the system, can be defined as

$$J_\mu = \frac{1}{4\pi} \partial_\nu (\partial_\mu a_\nu - \partial_\nu a_\mu). \tag{2.4}$$

Monopole loop and the loop of electric currents fall on the top of each other which can be proved by the fact that the magnetic charge can be obtained from the colour of an electric charge inside a three dimensional region Ω [33]. The (anti-) self-duality may be written as

$$F_{\mu\nu} = \pm \tilde{F}_{\mu\nu}, \tag{2.5}$$

where (+) sign is for the instanton and (−) sign is for the anti-instanton. This (anti-) self-duality survives the Abelian projection.

Gauge fixing and Abelian projection turns instantons into dyons [31, 32] with electric charge $e = \pm m$. Dynamical variables of the Abelian theory were 'photons', colour electrically charged particles and monopoles. Possible bound states of these particles can also be studied in Abelian gauge theory. Bound state [20–23] of monopole (monopolnium) has been investigated and it is used to solve the bound state of two dyons under the influence of externally applied magnetic field. In order to investigate the behaviour of the system (bound state of dyon) one can start with the quantum mechanical Hamiltonian [34]

$$H = -\frac{\hbar^2}{2m} \nabla^2 + \frac{i\hbar}{mc} (e\vec{A} - g\vec{B}) - \frac{iq\hbar}{mc^2} \dot{\varphi} + \frac{1}{2mc^2} [(e^2 - g^2)(A^2 - B^2) - 4eg\vec{A} \cdot \vec{B}] + q\varphi, \tag{2.6}$$

where q is charge on dyon given as

$$q = e - ig. \tag{2.6a}$$

For total field along Z-direction, $\vec{A}$ and $\vec{B}$ are given as

$$\left.\begin{aligned} A_x = -\tfrac{1}{2}H_y, A_y = \tfrac{1}{2}H_x, A_z = 0 \\ B_x = -\tfrac{1}{2}E_y, B_y = \tfrac{1}{2}E_x, B_z = 0 \end{aligned}\right\}, \tag{2.7}$$

where, $\vec{E}$ and $\vec{H}$ are electric and magnetic fields given as

$$\left.\begin{aligned} \vec{E} = -e\varphi + \tfrac{\partial \vec{A}}{\partial t} - \vec{\nabla} \times \vec{B} \\ \vec{H} = -g\varphi + \tfrac{\partial \vec{B}}{\partial t} - \vec{\nabla} \times \vec{A} \end{aligned}\right\}. \tag{2.8}$$

In this case,

$$\left.\begin{aligned} A^2 = \tfrac{1}{4}H^2 r^2 \sin^2\theta \\ B^2 = \tfrac{1}{4}E^2 r^2 \sin^2\theta \end{aligned}\right\}, \tag{2.9}$$

and

$$\left.\begin{aligned} \vec{A} \cdot \vec{\nabla} = \tfrac{1}{2}\vec{H}\tfrac{\partial}{\partial\phi} \\ \vec{B} \cdot \vec{\nabla} = \tfrac{1}{2}\vec{E}\tfrac{\partial}{\partial\phi} \end{aligned}\right\}, \tag{2.10}$$

The analysis of the transition is very important to understand the nature of the lines emitted. Total energy of the system is given as

$$E = E_0 + \frac{hm'}{4\pi mc}(eH + gE), \tag{2.11}$$

where $m' = m\cos\theta$ is the magnetic quantum number of the system of a dyon moving in the field of another dyon. Frequency of the lines emitted during the transition may be given as

$$\nu = \frac{E_2^{(0)} - E_1^{(0)}}{h} + \frac{\Delta m}{4\pi mc}(eH + gE), \tag{2.12}$$

which shows that the energy levels are modified due to the presence of magnetic charge on dyon and degeneracy of energy levels is lifted causing splitting of the energy levels in the usual way.

Motion of strongly correlated instantons and anti-instantons is influenced by the presence of external electric field also. Though it is very early to say anything about the experimental verification of these facts of such a system even than it is interesting to carry out the theoretical investigations using Hamiltonian approach.

Let us start with the Hamiltonian

$$\hat{H} = -\frac{\hbar^2}{2m}\hat{\nabla}^2 - \frac{\mathrm{Re}(q_i q_j^*)}{r} + \frac{\mathrm{Im}(q_i q_j^*)}{2mr^2} - |q|Z_E = \hat{H}_0 + \hat{H}_1, \tag{2.13}$$

where charge q is given by (2.6a). H_0 is the unperturbed Hamiltonian and H_1 is the perturbation part which is due to the influence of external electric field that is applied in z-direction.

The first order perturbation energy in ground state is

$$\begin{aligned}\lambda H^{(1)}_{100,100} &= -|q|E\iiint r\cos\theta\left(\frac{1}{\pi a_0^3}\right)e^{-2(\frac{r}{a_0})}r^2\sin^2\theta\,dr\,d\theta\,d\phi,\\ &= 0.\end{aligned} \tag{2.14}$$

This equation shows that no first order stark effect has been found in the ground state of instantons (anti-) instanton system. On introducing perturbation, the corresponding energy values are given as

$$\lambda E^{(1)} = 0, 0, +3a_0|q|E, -3a_0|q|E \tag{2.15}$$

which shows that two of the four degenerate states for $n = 2$ are unaffected by the electric field to the first order and other two form linear combinations ($\frac{\psi_{2,0,0}+\psi_{2,1,0}}{2}$) with extra energy $+3a_0|q|F$ in the electric field and ($\frac{\psi_{2,0,0}-\psi_{2,1,0}}{2}$) with extra energy $-3a_0|q|F$. This means that the dyonium in this state behaves as though it has a permanent moment of magnitude $3a_0|q|F$ which can be oriented in three different ways one state parallel to the field and two states perpendicular to the field with zero components along the field.

Effect of strong magnetic field on dyon (anti-) dyon system is rather important to understand the duality, which may confirm the theoretical existence of dyons. For this purpose strong magnetic field can be taken in such a way that the spin–orbit interaction term acts as perturbation and the term containing magnetic field as part of unperturbed Hamiltonian, then, the Hamiltonian for the system can be written as

$$\hat{H} = \hat{H}^{(0)} + \hat{H}', \tag{2.16}$$

where,

$$\hat{H}^{(0)} = \frac{\pi^2}{2m} + V(r) - \frac{\mu_{ij}(H+E)}{2m}(L_Z + 2S_Z), \tag{2.17}$$

is unperturbed Hamiltonian and

$$\hat{H}' = F(r)\hat{L}\cdot\hat{S}, \tag{2.18}$$

is the perturbation term which arises due to spin–orbit interaction. In this case the wave function is taken as Ψ_{μ,l',m_l,m_s} and the corresponding energy levels are obtained as

$$E_n - \frac{\mu_{ij}(H+E)}{2m}(m_l + 2m_s), \tag{2.19}$$

where

$$m_l = -1 \text{ to } +1 \quad \text{and} \quad m_s = \pm\frac{1}{2}. \tag{2.20}$$

Here E_n is the energy eigen value of the system given as

$$E_n = -2\alpha_{ij}^2 m\Big[(2n+1) + \{(2l+1)^2 + 4\mu_{ij}^2\}^{1/2}\Big]^{-2}, \tag{2.21}$$

where $n = 0, 1, 2, \ldots$.

In (2.21), α_{ij} and μ_{ij} are electric and magnetic coupling parameters of ith and jth generalized charges moving in the field of one another given as [34]

$$\left.\begin{aligned}\alpha_{ij} &= e_i e_j + g_i g_j\\ \mu_{ij} &= e_i g_j - g_i e_j\end{aligned}\right\}. \tag{2.22}$$

Now, for the two states given by wave function $\Psi_{(n,l,m_l,m_s=\pm 1/2)}$, $\hat{L}_Z$ and $\hat{S}_Z$ are well defined and expectation values for $\hat{L}_x$, $\hat{S}_x$, $\hat{L}_y$ and $\hat{S}_y$ are zero. Thus the perturbation is given by

$$\langle F(r)\hat{L}\cdot\hat{S}\rangle = \xi_{n,l,m_l,m_s}, \tag{2.23}$$

$$\begin{aligned}\xi_{n,l} &= \langle F(r)\rangle\\ &= \frac{\mathrm{Re}(q_i q_j^*)^2}{m^2\{2n^3 l(l+1/2)(l+1)\}} - \frac{\mathrm{Im}(q_i q_j^*)^4}{m^4c^4}\left\{\frac{3-5n^3(l+1/2)a_0^2}{n^5(l-1/2)(l+1/2)(l+3/2)a_0^4}\right\},\end{aligned} \tag{2.24}$$

and the energy levels are given as

$$E = E_n + m(E+H)(m_l + 2m_s) + \xi_{n,l,m_l,m_s}. \tag{2.25}$$

This equation represents the Paschen-Back effect for dyonium. It can be seen that the operators L^2, S^2, $\hat{L}_Z$ and $\hat{S}_Z$ have well defined values for each state. Since dyons are very strongly interacting particles [34–36] so that the applied magnetic field is required to be very high in order to get the splitting in spectral lines.

3 Interaction of Dyon with Spin-1/2 Particle in SU(2) Gauge Theory

Spontaneous breaking of Supersymmetry at mass scale $M_x \sim 10^{15}$ Gev reveals the presence of monopoles and dyons [37–43] and that if Supersymmetry breaks at a scale much less than M_x the monopole and dyon must form supermultiplets of bosons and fermions (spin-1/2 particles). As such, supersymmetry provides first realistic testing ground for the idea of fermion fractionalization and induced fermionic charge on monopole converting it to a dyon. Therefore the occurrence of fractionally charged dyon along with their superpartners in each charge state would lead to the possibility of observation of dyons. In order to study the quantal properties of these monopoles and dyons for their possible experimental observation, it became necessary to investigate the bound state of dyon with itself and also with the fermion.

For better understanding of the problem Dirac equation is to be carried out for fermion every where in the field of non-Abelian dyon. This problem has enormous potential importance in view of its link with baryon number non-conservation and the recent results [44] that an exact solution of Dirac equation describing the motion of a fermion in the field of a dyon is not possible due to the presence of a term in the potential of the system, which vanishes more rapidly than r^{-1}. Apart from ‘t Hooft Polyakov [5–10] non-Abelian monopole i.e. Bogomol’nyi–Prasad–Sommerfield monopoles [17–19] have attracted much attention in a variety of context in theoretical and mathematical physics [46, 47] and more recently in pure mathematics BPS monopoles provide a three dimensional example of topological solitons of Bogomol’nyi type; they are static, finite energy solutions of classical field equations and stable because their energy attains a lower topological bound, Bogomol’nyi bound.

Much progress in understanding of dynamics of such solitons has been made over the past decade using the idea of moduli space approximation [44]. Undertaking the study of monopoles and dyons in moduli space the extended structure of non-Abelian dyons and the structure for dyonic mass and electric and magnetic fields in the interior region of dyon has been analyzed [48, 49] and showed that when collective coordinates of monopoles are time dependent it acquires momentum and electric charge and become a moving dyon. The connection of moduli space of monopoles with normalized bosonic and fermionic zero modes has also been analyzed.

To study the interaction of fermion in the field of dyon let us write $\psi(t,\vec{x})$ for a four-component spinor which also transforms under the fundamental representation of the SU(2) isospin group, the $(1+4)$-dimensional Dirac equation for non-Abelian dyon in moduli space [45–47] in the temporal gauge $V_0=0$, may be written as

$$[-\{\Gamma^0\otimes\partial_t + c\Gamma^\mu\otimes D_\mu\} + M_0c^2]\psi = 0, \tag{3.1}$$

where M_0 is the mass of non-Abelian dyon and V_0 is the temporal part of the generalized four-potential $\vec{V}_\mu = V_\mu^a T_a$. The vector sign is denoted in the internal group space; $\mu = 0, 1, 2, 3, 4$ represents degree of freedom in the external space. The matrices T_a $(a = 1, 2, 3)$ are infinitesimal generators of the group SU(2) satisfying,

$$[T_a, T_b] = \varepsilon_{abc}T_c,$$

which can be expressed in terms of the Pauli matrices τ_a via $T_a = \frac{1}{2i}\tau_a$; dyonic generalized four-potential V_μ^a is defined as

$$V_\mu^a = A_\mu^a - iB_\mu^a. \tag{3.2}$$

For this system, specifically, we consider SU(2) gauge potential $\vec{V}_\mu$ $(\mu = 1, 2, 3, 4)$ on $R^4 = R^3\times R$ which are independent of X_4.

Standard Dirac γ matrices give five 4×4 complex matrices (Γ^0, Γ^μ) which can be written as

$$\Gamma^0 = \gamma^0, \qquad \Gamma^i = \gamma^i, \qquad \Gamma^4 = -i\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3. \tag{3.3}$$

ψ really transforms under a spinor representation of SO(1, 4) but we can think of it as an SO(1, 3) spinor by restricting to the Lorentz transformations in SO(1, 3) $\subset$ SO(1, 4) respecting the condition $x_4 = 0$. Dirac's equation for a fermion of charge e_1 moving in an external field of non-Abelian dyon in moduli space may be written as

$$[-\{\Gamma^0\otimes\partial_t + c\Gamma^\mu\otimes(\partial_\mu + e_1 g\vec{V}_\mu)\} + M_0c^2]\psi = 0. \tag{3.4}$$

On multiplying by Γ^4, (3.3) reduces to

$$[-\Gamma^4\gamma^0\otimes\partial_t - c\Gamma^4\gamma^i\otimes D_i - c(\Gamma^4)^2\otimes(\partial_4 + e_1 g\vec{V}_4) + \Gamma^4 M_0c^2]\psi = 0. \tag{3.5}$$

Using $\Gamma^4 = l_4$, $D_4 = \phi = c(\partial_4 + e_1 g\vec{V}_4)$ in (3.3) and (3.5) gives

$$\left[-\begin{bmatrix} l_2 & 0\\ 0 & -l_2\end{bmatrix}\otimes\partial_i - c\begin{bmatrix} 0 & \sigma_i\\ -\sigma_i & 0\end{bmatrix}\otimes D_i - \begin{bmatrix} l_2 & 0\\ 0 & -l_2\end{bmatrix}\otimes\phi + \begin{bmatrix} l_2 & 0\\ 0 & -l_2\end{bmatrix}M_0c^2\right]\psi = 0$$

or

$$i\hbar\frac{\partial}{\partial t}\begin{bmatrix} l_2 & 0\\ 0 & -l_2\end{bmatrix}\psi = \left[-c\begin{bmatrix} 0 & \sigma_i\otimes D_i\\ -\sigma_i\otimes D_i & 0\end{bmatrix} - \begin{bmatrix} l_2\otimes\phi & 0\\ 0 & -l_2\otimes\phi\end{bmatrix}\right.$$
$$\left. + \begin{bmatrix} l_2 & 0\\ 0 & -l_2\end{bmatrix}M_0c^2\right]\psi = 0. \tag{3.6}$$

The relativistic energy of the particle includes its rest energy M_0c^2. This must be excluded in arriving at the non-relativistic approximation, and therefore ψ is replaced by a function ψ', defined as follows:

$$\psi = \psi' e^{-iM_0c^2t/\hbar}.$$

Equation (3.6) is then written as

$$[i\hbar(\partial/\partial t) + M_0c^2]\begin{bmatrix} l_2 & 0 \\ 0 & -l_2 \end{bmatrix}\psi' = \left[-c\begin{bmatrix} 0 & \sigma_i \otimes D_i \\ -\sigma_i \otimes D_i & 0 \end{bmatrix} - \begin{bmatrix} l_2 \otimes \phi & 0 \\ 0 & -l_2 \otimes \phi \end{bmatrix} + \begin{bmatrix} l_2 & 0 \\ 0 & -l_2 \end{bmatrix} M_0c^2\right]\psi'.$$

On substituting $\psi' = \binom{\xi}{\eta}$, where ξ and η are two-component functions, this equation reduces to

$$\begin{bmatrix} [i\hbar(\partial/\partial t) + M_0c^2]\xi \\ -[i\hbar(\partial/\partial t) + M_0c^2]\eta \end{bmatrix} = \begin{bmatrix} -c\sigma_i \otimes D_i\eta \\ c\sigma_i \otimes D_i\xi \end{bmatrix} + \begin{bmatrix} -l_2 \otimes \phi\xi \\ -l_2 \otimes \phi\eta \end{bmatrix} + \begin{bmatrix} M_0c^2\xi \\ M_0c^2\eta \end{bmatrix}. \tag{3.7}$$

Equation (3.7) gives

$$[i\hbar(\partial/\partial t) + l_2 \otimes \phi]\xi = -c\sigma_i \otimes (\partial_i + e_1 g\vec{V}_i)\eta, \tag{3.8}$$

$$-[i\hbar(\partial/\partial t) + l_2 \otimes \phi + 2M_0c^2]\eta = c\sigma_i \otimes (\partial_i + e_1 g\vec{V}_i)\xi. \tag{3.9}$$

In the first approximation, only the term $2M_0c^2\eta$ is retained on the left hand side ode (3.9), which gives

$$\eta = -\left(\frac{1}{2M_0c}\right)\sigma_i \otimes (\partial_i + e_1 g\vec{V}_i)\xi. \tag{3.10}$$

Substituting (3.10) into (3.8) gives

$$i\hbar\frac{\partial\xi}{\partial t} = \left[\frac{1}{2M_0}(\vec{P}_l + e_1 g\vec{V}_i)^2 - l_2 \otimes \phi - \frac{e_1 g\hbar}{2M_0}\sigma_i \otimes \mathrm{curl} V_i\right]\xi = \hat{H}\xi \tag{3.11}$$

where $\partial_i = \vec{P}_i$ is the momentum of a non-Abelian dyon.

This is the Pauli equation for a fermion in the field of non-Abelian dyon in moduli space. It has the following extra spin contribution in the energy gained by fermion while moving in the field of non-Abelian dyon;

$$E' = -\frac{e_1 g\hbar}{2M_0}[\sigma_i \otimes \mathrm{curl}\vec{V}_i]. \tag{3.12}$$

This equation can also be written as

$$E' = -\mu_{D'}(\sigma_i \otimes \mathrm{curl}\vec{V}_i), \tag{3.13}$$

where

$$\mu_{D'} = \frac{e_1 g\hbar}{2M_0}, \tag{3.14}$$

is defined as Bohr magneton for the system and

$$\mu_D = \mu_{D'}\sigma_i, \tag{3.15}$$

as generalized spin momentum of this system. Consequently, extra-energy term in the Hamiltonian may be interpreted as the energy of interaction of the spin moment of a fermion with the vector field, resulting from the space rotation of generalized four-potential $\{\vec{V}_\mu\}$. The third component of the generalized spin moment operator for fermion may be written as:

$$(\mu_D)_3 = \frac{e_1 g\hbar}{2M_0}\sigma_3. \tag{3.16}$$

The eigen value of which is

$$\pm\frac{e_1 g\hbar}{2M_0} = \pm\mu_{D'}. \tag{3.17}$$

4 Relativistic Corrections and Angular Momentum Operators of Fermion–Dyon (Generalized EM Field of Non-Abelian Dyon) System in Moduli Space

As such, it could not be possible so far to solve Dirac equation completely for dyon–fermion bound states. In light of these facts, it becomes necessary to develop the alternative approach for studying fermion–dyon dynamics in terms of Green function of U(1) Higgs model expanding in terms of dyonic harmonics in topological field theories. This difficulty has been resolved partially by some authors [34, 45–47] by using moduli space for BPS monopoles. Later on moduli space approximation has been used to study the bound states of BPS monopoles and dyons. In this section, we study the motion of a fermion in the generalized electromagnetic field of non-Abelian dyon in moduli space by retaining terms up to those of the order of v^2/c^2. Substituting $\vec{V}_i = 0$ and $E = i\hbar(\partial/\partial t)$ in (3.8) and (3.9), we get

$$(E + l_2 \otimes \phi)\xi = -c\sigma_i \otimes \vec{P}_i\eta, \tag{4.1}$$

$$-(E - l_2 \otimes \phi + 2M_0c^2)\eta = c\sigma_i \otimes \vec{P}_i\xi. \tag{4.2}$$

Function η up to the first order in $(E - l_2 \otimes \phi)/2M_0c^2$ can be calculated by (4.2). Substituting the value

$$\eta = -\frac{1}{2M_0c}\left[1 - \frac{(E - l_2 \otimes \phi)}{2M_0c^2}\right](\sigma_i \otimes \vec{P}_i)\xi$$

in (4.1), an equation containing only one two-component function can be obtained as

$$(E + l_2 \otimes \phi)\xi = \frac{1}{2M_0}(\sigma_i \otimes \vec{P}_i)\left[1 - \frac{(E - l_2 \otimes \phi)}{2M_0c^2}\right](\sigma_i \otimes \vec{P}_i)\xi, \tag{4.3}$$

which on simplification gives the following expression for energy operator (Hamiltonian) in the first approximation;

$$\hat{H} = \left(\frac{1}{2M_0}\right)\left[1 - \frac{(E - l_2 \otimes \phi)}{2M_0c^2}\right]\vec{P}_i^2 - l_2 \otimes \phi - \left(\frac{i\hbar}{4M_0^2c^2}\right)[\vec{\nabla}(l_2 \otimes \phi) \otimes \vec{P}_i]$$
$$+ \left(\frac{\hbar}{4M_0^2c^2}\right)[\sigma_i \otimes \{\vec{\nabla}(l_2 \otimes \phi)\} \times \vec{P}_i]. \tag{4.4}$$

In order to derive the expression for Hamiltonian in second approximation, we use another function χ instead of ξ given as

$$\chi = \hat{u}\xi,$$

the normalization of which, upto second order, leads to the following value of factor $\hat{u}$,

$$\hat{u} \approx 1 - \left(\frac{\vec{P}_i^2}{8M_0^2c^2}\right).$$

Using this value of $\hat{u}$ (and hence of χ), we get the following relativistic expression for corresponding Hamiltonian upto terms of the order of v^2/c^2;

$$\begin{aligned}\hat{H} &= \left[1 + \left(\frac{\vec{P}_i^2}{8M_0^2c^2}\right)\right]\hat{H}\left[1 - \left(\frac{\vec{P}_i^2}{8M_0^2c^2}\right)\right] \\ &= \left[\left(\frac{\vec{P}_i^2}{2M_0}\right) - (l_2 \otimes \phi)\right] + \left[\left(\frac{\hbar^2}{8M_0^2c^2}\right)\vec{\nabla}^2(l_2 \otimes \phi)\right] - [(E - l_2 \otimes \phi)^2/2M_0c^2] \\ &\quad + \left(\frac{\hbar^2}{4M_0^2c^2}\right)[\sigma_i \otimes \{\vec{\nabla}(l_2 \otimes \phi)\} \otimes \vec{P}_i], \end{aligned} \tag{4.5}$$

where $\hat{H}_0$ corresponds to the non-relativistic term of the Hamiltonian, while $\hat{H}_1$ is the relativistic correction term to the Hamiltonian various parts of which arise due to different relativistic interaction. The quantity $\hat{H}_1$ is called contact interaction operator, analogous to the term introduced by Darwin [50] for electronic case. $\hat{H}_2$ is the relativistic correction term due to the dependence of kinetic energy on momentum. Finally,

$$\hat{H}_3 = \frac{\hbar[\sigma_i \otimes \{\vec{\nabla}(l_2 \otimes \phi)\} \times \vec{P}_i]}{4M_0^2c^2}, \tag{4.6}$$

is the so called spin-orbit interaction operator.

In a spherically symmetric field

$$\vec{\nabla}\phi = \frac{\vec{r}}{r}\frac{d\phi}{dr},$$

(4.6) gives the spin–orbit interaction operator for the motion of a spin-$1/2$ particle in a spherically symmetric field;

$$\hat{H}_3 = \frac{d}{dr}(l_2 \otimes \phi)\frac{(\hat{S} \otimes \hat{L})}{2M_0^2c^2r}. \tag{4.7}$$

Where $\hat{L} = \vec{r} \times \vec{P}_i$ is the orbital angular momentum operator and $\hat{S} = 1/2\hbar\sigma_i$ is spin angular momentum operator for the system. This expression clearly demonstrates that besides the contribution of Higgs field, the interaction of spin and orbital angular momentum of moving fermion also contributes to the energy operator.

5 Discussion

Equation (2.1) is the angular momentum operator and (2.2) represents energy eigen value of the system (Abelian) giving degeneracies for the same. In this section, we have restricted ourselves to a particular bound state energy level ($E < 0$) of monopole and confine to the invariant subspace (of the full space) which corresponds to the eigen value E. Conserved electric current of the system has been given as (2.4) and (anti-) self-duality is given by (2.5) that survives the Abelian projection. Quantum mechanical Hamiltonian of bound state of dyon is given by (2.6) which gives frequency of the lines emitted during the transition. This equation shows that the energy levels are modified due to the presence of magnetic charge on dyon and degeneracy of energy levels is lifted causing splitting of the energy levels in the usual way. First order perturbation energy in ground state is given by (2.14) showing that no first order stark effect has been found in this state. Energy eigen value corresponding to the perturbation has been given by (2.15) explains that dyonium in this state behaves as it has a permanent moment of magnitude $3a_0|q|F$. To understand the effect of strong magnetic field on dyonium a Hamiltonian including spin–orbit interaction has been written as (2.17) which gives the energy levels for the system given by (2.25). Dirac equation for non-Abelian dyon in moduli space is given by (3.1). Standard Dirac matrices and Dirac equation for a fermion in an external field of non-Abelian dyon are given by (3.3) and (3.4) respectively. Pauli equation for a fermion in the field of non-Abelian dyon in moduli space is given by (3.11) shows extra spin contribution in the energy gained by the fermion. Bohr magneton and the third component of the generalized spin moment operator for fermion have been written as (3.14) and (3.16) respectively. Equation (4.5) is the relativistic Hamiltonian, for fermion in the field of non-Abelian dyon, different parts of which arises due to different relativistic interaction. Expressions for energy operator in first and second order approximations are given by (4.4) and (4.5) respectively which contains non-relativistic term as well as relativistic corrections. Hamiltonian of this system has been shown in terms of Higgs potential instead of scalar potential in my forthcoming paper [51, 52] in Abelian as well as in non-Abelian gauge theories due to moduli space approximation.

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